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REVISION OF ALGEBRAIC CONCEPTS

Algebra is based on man made agreements on conventions, symbols and rules which are based on logic.

Number systems

Natural numbers $N = \{1, 2, 3, ...\}$

Natural numbers plus 0 $N_0 = \{0, 1, 2, 3, ...\}$

Integers $Z = \{... -2, -1, 0, 1, 2, ...\}$

Rational numbers $Q = \left\{ \frac{p}{q}, \text{ with } p,q \in Z,\ q \neq 0 \right\}$, e.g. -4, 0, $\frac{-3}{5}$, $0,\dot{3}$

Irrational numbers $I = \{ x \in R \text{ and } x \notin Q \}$ e.g. π, $\sqrt{2}$, e

Real numbers $R = \{ x \in Q \text{ or } x \in I \}$

It follows that all the sets above are subsets of the real numbers i.e.
$N \subset N_0 \subset Z \subset Q \subset R$ and $I \subset R$

Furthermore $Q \cup I = R$ and $Q \cap I = \varnothing$

Basic properties of real numbers

Properties of the number 0:

$p \cdot 0 = 0 \cdot p = 0$

$\frac{0}{p} = 0$ with $p \neq 0$

$\frac{p}{0}$ is undefined.

If $p \cdot q = 0$, then $p = 0$ or $q = 0$.

If $\frac{a}{b} = 0$ then $a = 0$ and $b \neq 0$

Order of operations

Priority	Operation	Explanation
1ste	() brackets	From inner to outer
2de	Exponents and roots	Regard as brackets
3de	"of"	Replace with $\times$-sign
4de	$\times$ and/or $\div$	Equivalent priority
5de	$+$ and/or $-$	Equivalent priority

Properties of operations with real numbers

Let $p,q,r \in R$:

Closed sets: $p+q \in R$ and $pq \in R$

Commutative: $p+q=q+p$ and $pq=qp$

Associative: $(p+q)+r=p+(q+r)$ and $(pq)r=p(qr)$

Identity: $p+0=p$ and $p\cdot 1=p$

Inverse: $p+(-p)=0$ and $p\cdot\frac{1}{p}=1$

Distributive: $p(q+r)=pq+pr$

The operation of absolute value

The absolute value of any number means "to make number positive".

Formulation of absolute number in words:

$|\text{positive number}|$ = positive numberl

$|\text{negative number}|$ = –(negative number)

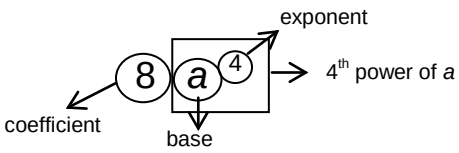
$|0|$ = 0

Properties of exponents

Notation

Definition: $\underbrace{(a\times a\times a\times ...\times a)}_{n \text{ times}}=a^n$

Eg: If $(2a\times 2a\times 2a)=8a^3$, then the terminology is:



Rules of operations with exponents

$a^0=1$

$a^1=a$

$a^m\cdot a^n=a^{m+n}$

$(a^m)^n=a^{mn}$

$\frac{a^m}{a^n}=a^{m-n}$ with $a^n\neq 0$

$a^{-n}=\frac{1}{a^n}=\left(\frac{1}{a}\right)^n$ and $a\neq 0$

Definition of the natural base of exponents

Any positive number can serve as the base of the exponential function. A number that is frequently used as a base is the number e.

Consider the expression $(1+\frac{1}{n})^n$ and determine what will happen if $n \to \infty$, i.e. if n becomes very large.

n	$\left(\frac{n+1}{n}\right)^n$
1	
10	
100	
1000	
1000000	

The number e is defined as: $e = \lim_{n\to\infty} (1+\frac{1}{n})^n$.

Properties of logarithms

Definition of a logarithm

If $a^x = y$ and the subject of the formula must be changed to x, then $x = \log_a y$.

Rules of operations with logarithms

$\log_a 1 = 0$

$\log_a a = 1$

$\log_a \frac{1}{a} = -1$

$\log_a mn = \log_a m + \log_a n$

$\log_a \frac{m}{n} = \log_a m - \log_a n$ en $n \neq 0$

$\log_a b^n = n \log_a b$

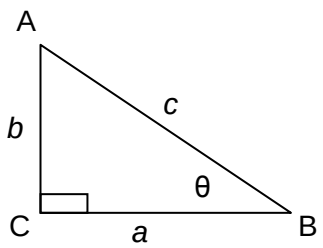
REVISION OF TRIGONOMETRIC CONCEPTS

Definition of trigonometric ratios

$\sin\theta = \frac{b}{c}$ $\csc\theta = \frac{c}{b}$

$\cos\theta = \frac{a}{c}$ $\sec\theta = \frac{c}{a}$

$\tan\theta = \frac{b}{a}$ $\cot\theta = \frac{a}{b}$



Trigonometric identities

$\sin\theta = \dfrac{1}{\csc\theta}$ $\cos\theta = \dfrac{1}{\sec\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$

Trigonometric formulas

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\sin 2A = 2\sin A\cos A$

EXERCISE

1. Complete the following statements using $+$, $-$, $\times$, and/or $\div$:

(a) The set of natural numbers is closed with respect to …….

(b) The set of integers is closed with respect to ……

(c) The set of rational numbers is closed with respect to …….

2. Simplify:

(a) $\dfrac{0}{0}$ = …… (b) $\dfrac{0}{1000}$ = ……

(c) $\dfrac{1000}{0}$ = …… (d) $\dfrac{1000}{1000}$ = ……

(e) (4)(3 + 5) = ...

(f) (4)(3 × 5) = ...

(g) $\dfrac{-3t^2 - 6ht - 3h^2 + 5t + 5h - 2 + 3t^2 - 5t + 2}{h}$ = ...

.....

(h) $4 - 3\left|(-2)(3)\right| + \dfrac{5^3}{25} - \dfrac{12}{2 + \frac{2}{5}} - \sqrt{5^2 - 3^2}$ = ...

.....

(i) $\sqrt[4]{48}$ = ...

3. Simplify:

(a) $(2x^2)(2x^2)(2x^2)$ = ...

(b) $2x^2 + 2x^2 + 2x^2$ = ...

(c) $(4^2)^{-1}\,(2^{-1})^3$ = ...

(d) $(-x^{-1})(2x^3)(-3x^{-3})$ = ...

(e) $(x^2 + y^2)(x^2 - y^2)$ = ...

(f) $(x + 2)(x^2 + x + 4)$ = ...

.....

(g) $\dfrac{2}{3x} - \dfrac{4}{5x} + \dfrac{3}{2x}$ = ...

.....

(h) $1 - \dfrac{x-4}{4x} - \dfrac{x+1}{2x}$ = ...

.....

(i) $\dfrac{4x+1}{2x} - \dfrac{x^2-2}{3x^2}$ = ...

.....

(j) $\dfrac{-8x^2}{y^2} \times \dfrac{-5y}{4x} \div \dfrac{10x}{y^2}$ = ...

.....

(k) $\dfrac{x+2}{x}$ of $\dfrac{3x^2}{(x+2)^2} \div \dfrac{x}{3}$ = ...

.....

4. Simplify:

(a) $|2|$ = ...

(b) $|-2|$ = ...

(c) $|2-7| - |7-2|$ = ...

(d) $\left|\sqrt{5}-2\right| =$ ……

(e) $\left|2-\sqrt{5}\right| =$ .....

(f) $\left|2^{-1}\right| =$ ……..

5. Simplify:

(a) $\dfrac{2^3+2^2}{2^5} =$ .....

(b) $\dfrac{x^3+x^2}{x^2} =$ .....

(c) $\dfrac{x^5+x^5}{x^2+x^2} =$ .....

(d) $\dfrac{3^x.9^{x+1}}{27^{x+2}} =$ .....

.....

(e) $\dfrac{2^{x+3}-2^{x+2}}{2^{x+4}} =$ .....

.....

(f) $\dfrac{2^{x+2}+2^{x+1}}{2^{x+2}+2^{x+1}} =$ .....

.....

6. Rewrite as a logarithm:

(a) $8^{\frac{2}{3}}=4$ …… (b) $e^{0,33647}=1,4$ ……..

7. Rewrite as a power:

(a) $\log 5 = 0,6990$ …… (b) $\ln 4 = 1,386$ ……

8. Simplify:

(a) $\log_2 3+\log_2 5 =$ .....

(b) $\log_2 3+\log_3 2 =$ .....

(c) $\log_2 \sqrt[3]{2} =$ .....

(d) $\log_5 \dfrac{1}{25} =$ .....

(e) $\log_5 (5\sqrt{5})^5 =$ .....

9. Simplify:

(a) $\dfrac{\sin^2 x + \sin x}{\sin x}$ = ...

.....

(b) $\dfrac{1}{2}\sin\dfrac{x}{2}\cos\dfrac{x}{2}$ = ...

.....

(c) $\dfrac{\sin x}{\sin^2 x}$ = ...

(d) $\dfrac{\sin^2 x + \cos^2 x}{\cos x}$ = ...

.....

(e) $\dfrac{\sec x - \cos x}{\sec x}$ = ...

.....

10. Change the subject of the formula to the symbol given in brackets:

(a) $I = \dfrac{nE}{R + nr}$ (n)

.....

.....

.....

.....

(b) $A = \pi r^2 + \pi rs$ (r)

.....

.....

.....

.....

(c) $V = \dfrac{\pi(D^3 - d^3)}{6}$ (d)

.....

.....

.....

.....

(d) $I = I_0 e^{-\lambda t}$ (t)

.....

.....

.....

.....

(e) $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ (T_2)

.....

.....

.....

.....

FUNCTION ANALYSIS

1. Revision of function concept

1.1 Introduction

We encounter **relations** in everyday life where objects are paired together according to a relationship or a rule, e.g. each child has two parents, each town has a postal code, when running the Comrades the distance travelled will depend on the time elapsed. Often, particularly in science, business and engineering, these relationships involve numbers that are paired together according to a mathematical formula. When a relationship between two numbers is unique, it is called a **function**. The idea of a function is one of the unifying or big ideas of mathematics.

1.2 Formal definition of a function

A **relation** involves two sets where the elements of one set, called the **domain**, is paired (or mapped to) the elements of the second set, called the **range**. The statements "*We need 6 eggs for our kitchen*" or "*I have R3-00*" do not mean much in isolation, but by adding the relationship "*The eggs cost R2-00 per dozen"* one can decide to buy the eggs or not. However, a shop cannot have a notice saying that 6 eggs cost R2-50 and R3-75, because one cannot have two prices for one item. When a relation reflects a unique relationship it is called a **function**. A relation that is not a function is an example of a one-to-many mapping.

DEFINITION OF A FUNCTION

A **function** is a rule that assigns to each member in one set (the **domain**) exactly **one** member from another set (the **range**).

There are two types of functions as illustrated in the following example. Assume a promotion reads: "*Buy 9 eggs and get one free*". This in effect means that you pay the same price for 9 and for 10 eggs. Such a mapping is called a **many-to-one** pairing. Hence one can have the same price for different amounts, but not two prices for the same amount. Keeping this in mind, one can restrict the definition of a function to allow the same condition to apply to the range. In such a case we have a **one-to-one** pairing.

DEFINITION OF A ONE–TO–ONE FUNCTION

A **function** *f* is said to be **one–to–one** if every number in the range is paired with one and only one number in the domain.

Note that the rule does not have to be a mathematical formula, although in our work that will mostly be the case. Functional relationships can be represented in various ways, e.g. an example from real life, in a table or a graph, as a fromula or an ordered pair.

We usually depict the rule or relationship with an *f,* the the member of the domain by an *x* (**independent variable)** and the member of the range by a *y* (**dependent variable**). Remember that *f*(*x*) in this context means "*f* evaluated at *x*" and not "*f* multiplied by *x*".

Exercise

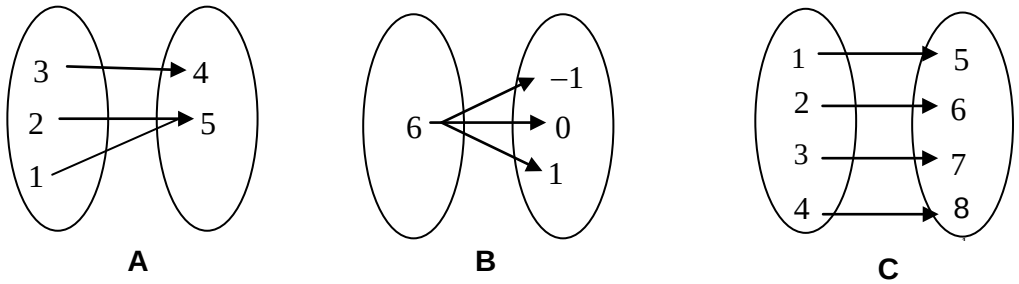
1. The domain of a certain function is {*x* / –2 < *x* < 2 and *x* is an integer}. The function is described by the rule: "*To each number in the domain assign its cube value."*

(a) Give the domain of the function: { …….}

(b) Write the function in function notation: *f*(*x*) = ……

(c) Give the range of the function: {……..}

2. Determine which of the following assignments represent **functions**.



3. Which of the functions in question 2 is one-to-one functions? ……

4. If A = {–1, 0, 1, 2, 3} is the domain and B = {–3, –2, –1, 0, 1, 2, …, 9, 10} is the range, write the following functions in terms of number pairs:

(a) $\{(x;y) \mid y = x, x \in A,\ y \in B\}$ …….....

(b) $\{(x;y) \mid y = x^2, x \in A, y \in B\}$ ……

5. Specify if the functions in no. 4 is **one–to–one** or **more–to–one** functions:

(a) …….

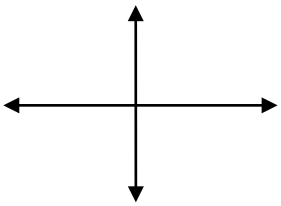
(b) …….

6. Give the domain and range of the following functions:

(a) $\{(x;y) \mid y = (x-1)^2 - 4,\ x \in \mathrm{R} \text{ en } y \in \mathrm{R}\}$

D_f = …….

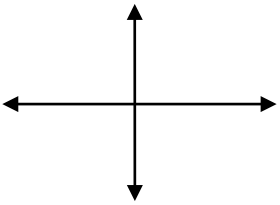
W_f = ……...



(b) $\{(x;y) \mid y = -(x-1)^2 - 4,\ x \in \mathrm{R} \text{ and } y \in \mathrm{R}\}$

D_f = ……

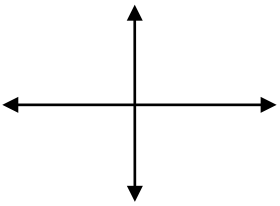
W_f = ……...



(c) $\{(x;y) \mid y = -\frac{8}{x},\ x \in \mathrm{R} \text{ and } y \in \mathrm{R}\}$

D_f = ……

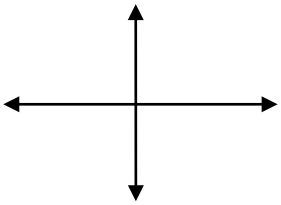
W_f = ……...



(d) $\{(x;y) \mid y = \sqrt{x-2},\ x \in \mathrm{R} \text{ and } y \in \mathrm{R}\}$

D_f = ……

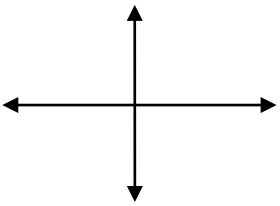
W_f = ……...



(e) $\{(x;y) \mid y = 2^x\ \ x \in \mathrm{R} \text{ and } y \in \mathrm{R}\}$

D_f = ……

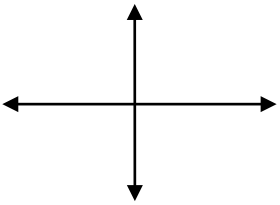
W_f = ……...



(f) $\{(x;y) \mid y = \log_2 x,\ x \in \mathrm{R} \text{ and } y \in \mathrm{R}\}$

D_f = ……

W_f = ……...



1.3 Inverse functions

In many applications of functions it is desirable to "reverse" the working of a function. This means that we interchange the "x" and "y" values so that the domain and range of the function switch places. We call this "reverse" function the **inverse function** of f and it is indicated by the notation f^{-1}. The graph of the inverse of a function is a mirror image of the graph of the original function with respect to the line $y = x$. From the definition of a function, it follows that the inverse of a function will only be a function if the original function was a on-to-one function.

Note: f^{-1} does not in general mean $\frac{1}{f}$.

Exercise

7. Determine the inverse of each of the following functions

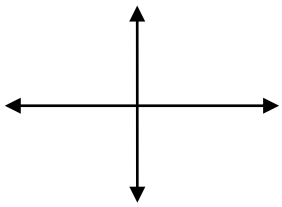
(a) $f(x) = 2x + 1$

.....

.....

.....

.....



(b) $f(x) = \sqrt{x - 1}$

.....

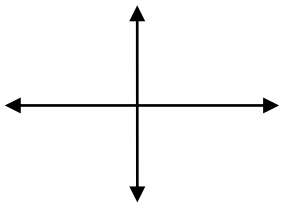
.....

.....

.....

.....

.....



1.4 Evaluating functions

The value of a function can be determined for any number in the domain by substituting the value of *x* into the expression representing the function.

Exercise

8. For each function given below, determine $f(0)$, $f(2)$ and $f(x+h)$:

(a) $f(x) = x^2 + 2x + 4$

.....

.....

.....

(b) $f(x) = \dfrac{1}{3x}$

.....

.....

.....

.....

1.5 Algebra of functions

Suppose *f* and *g* are functions, then new functions can be constructed by addition, subtraction, multiplication and division of the original functions. In general the domain the new function is the intersection of the domains of the original functions., but in the case of division the domain will exclude values for which function *g* is zero.

Exercise

9. If $f(x) = 2x^2$ and $g(x) = 3x + 5$ determine:

(a) $(f + g)(x)$ =..... ……

(b) $(g + f)(x)$ =..... ……

(c) $(f - g)(x)$ = ...……

(d) $(g - f)(x)$ = ...……

(e) $(fg)(x)$ = ...……

(f) $(gf)(x)$ = ...……

(g) $\left(\frac{f}{g}\right)(x) =$ .....……

(h) $\left(\frac{g}{f}\right)(x) =$ .....……

(i) What can you deduce about the commutative property of these operations from the answers in question 9

.....

.....

10. If $f(x) = 2x^2$, $a = 3$ and $x = 2$, determine:

(a) $a[f(x)] =$ ……. .....

(b) $f(ax) =$ …….....

(c) Make a conjecture based on these results.

.....

(d) Apply this conjecture to a few specific functions

.....

.....

.....

11. If $f(x) = \sqrt{x}$, $a = 9$ and $x = 16$, determine:

(a) $f(a + x) =$ ..... ……

(b) $f(a) + f(x) =$ ..... ……

(c) Make a conjecture based on these results.

.....

(d) Apply this conjecture to a few specific functions

.....

.....

.....

1.6 Composite functions

A new function can also be formed by combining two functions. This means that we operate on *x* with the first function, then use the answer as the *x*-value for the second function. The combination of two functions is not commutative, this means that the order in which one operates is important.

COMPOSITE FUNCTIONS

If *f* and *g* represent functions, then $(f \circ g)(x) = f(g(x))$ for all *x* in the domain of *g* such that $g(x)$ is the domain of *f*.

Example:

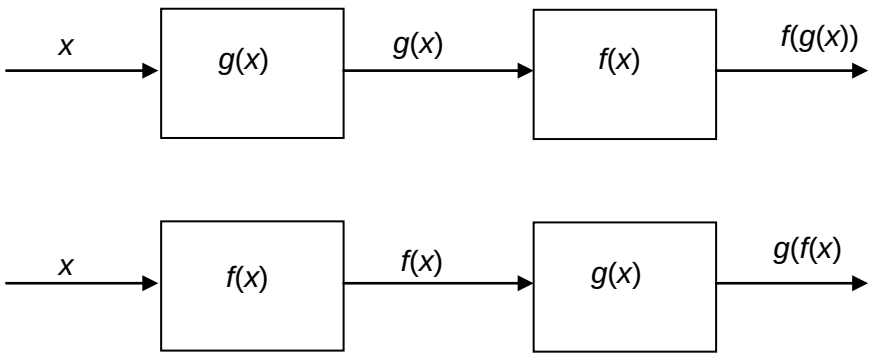
If $f(x) = 2x$ and $g(x) = 3x + 5$, find the composite functions.

<u>Solution</u>: $(f \circ g)(x) = f(g(x) = f(3x + 5) = 2(3x + 5) = 6x + 10$

$(g \circ f)(x) = g(f(x) = g(2x) = 3(2x) + 5 = 6x + 5$

It follows that the composition of two functions is not commutative.

<u>Schematic representation</u>:



Exercise

12. If $f(x) = x^2$ and $g(x) = \sqrt{x+2}$, determine:

(a) $(f \circ g)(x)$ = ...

.....

(b) $(g \circ f)(x)$ = ...

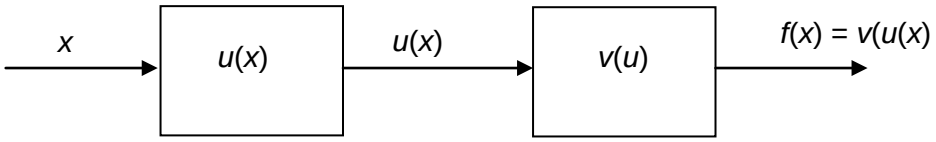
.....

(c) $(g \circ f)(2)$ =.....

When applying the chain rule to determine the derivative of a composite function, one first has to decompose the function. The notation that we use is similar to the notation that you will use when differentiating using the chain rule, i.e. u(x), v(u), etc.

Example:

If $f(x) = (2x+1)^2$, decompose the composite function into two separate functions $u(x)$ and $v(u)$.



<u>Solution</u>: $u(x) = 2x + 1$ and $v(u) = u^2$

Exercise

13. Decompose the following composite functions:

(a) $f(x) = \sqrt{3x-2}$

.....

(b) $f(x) = (2x+1)^8$

.....

(c) $f(x) = \dfrac{1}{(x+1)^2}$

.....

.....

.....

(d) $f(x) = 2^{3x}$

.....

(e) $f(x) = \log(2x + 2)$

.....

(f) $f(x) = \cos(x^2)$

.....

(g) $f(x) = \cos^2 x$

.....

(i) $f(x) = \cos(\sin x)$

.....

2. Graphs of linear and quadratic functions

2.1 Introduction

In mathematics there are in general two ways to approach the graph of a function: in functional analysis the problem is to draw the graph of the given formula; in applied mathematics the problem is to find a formula that will describe the graph. The latter process is called modelling. In this paragraph we focus on the first approach.

Functions can be represented graphically by sketching the number pairs (x,y) in the Cartesian plane. Usually there are an infinite number of points, but one only has to sketch a few points until the general behaviour of the graph becomes clear. The graph of a function can consist of discrete points or be continuous.

While at school you would have realised that certain groups of functions shows similar properties. A function of the form $f(x) = ax + b$ has a straight line graph , while the graph of the function $f(x) = \sin x$ shows a characteristic wave pattern. When sketching graphs one does not always have to construct a table of (x,y)–pairs to find the behaviour of the graph, one can use the common properties of the function group to draw the graph.

2.2 Graph of linear function

A function of the form $f(x) = mx + c$, where m and c are real numbers, is called a linear function. The graph of a linear function is a **straight line**. To determine the x–intercept one puts $y = 0$ and to determine the y–intercept one puts $x = 0$ in the equation of the straight line.

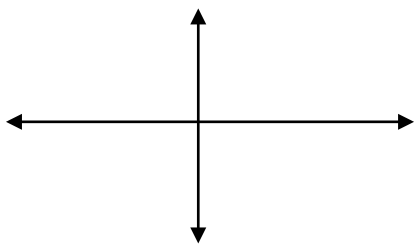
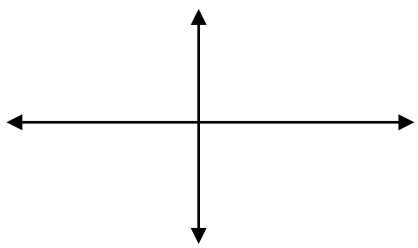
$y = x$ is called the **identity function**, because $f(x)$ is always equal to x. The graph of this function is a straight line through the origin with slope of 1. This line makes an angle of 45° with the x–axis. We use this function to find the inverse of any function.

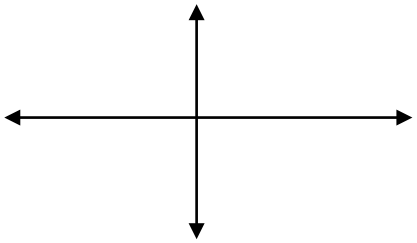
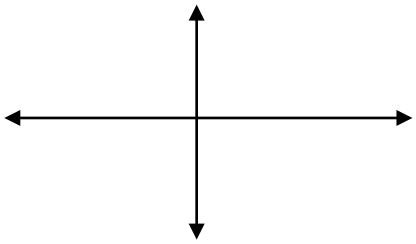
$y = 2$ is a **constant function** and its graph is a horizontal line that cuts the y–axis at 2.

$x = 2$ is not a function, but its graph is a vertical line that cuts the x–as at 2.

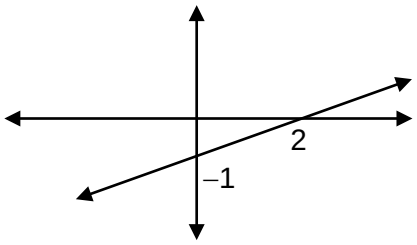
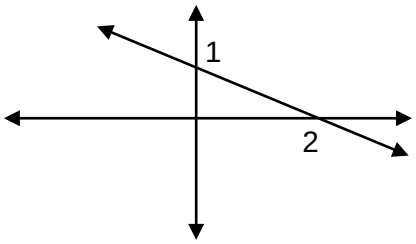
Exercise

1. Draw sketch graphs of the following relations:

(a) $f(x) = -1$ 	(b) $x = -1$ 

(c) $f(x) = -x + 1$ 	(d) $f(x) = 3x - 1$ 

2. Determine the equations of the following graphs:

(a) 	(b) 

2.3 Graph of quadratic function

The general equation of a **quadratic function is** $f(x) = ax^2 + bx + c$. The graph of this function is in the form of a parabola.

Graph of $f(x) = ax^2 + bx + c$

Determining the *y*-intercept:

To determine the *y*–intercept, one substitutes the *x*–value by zero:

$f(0)$ = …….

Determining the *x*-intercepts:

To determine the *x*–intercepts, let the *y*–value be zero:

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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Conclusions on the nature of the roots:

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Determine the turning points and the symmetry axis

Use completion the squares to write the equation in the form $y = a(x-p)^2 + q$.

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Symmetry–axis: $x = -\frac{b}{2a}$ Turning point: $y = \frac{4ac - b^2}{4a}$

Exercise

3. What is the relationship between the formula for the x-intercepts of a quadratic equation and the equation of the symmetry–axis?

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4. Use this relationship to deduce a new formula for the x-coordinate of the turning point of a quadratic function.

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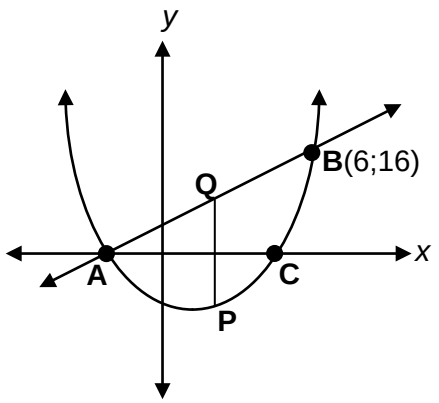
Exercise

5. The figure shows the graphs of the function $y = x^2 - 2x - 8$ and a straight line that goes through the points A on the x–as and B(6; 16). Q and P are points between A and B with Q on straight line and P on parabola so that QP is parallel to the y–axis.

(a) Determine the coordinates of A and C.



(b) Determine the equation of AB.

(c) For which value(s) of x will PQ be equal to 7 units?

(d) Determine the maximum value of the length of PQ.

(e) Use the graphs to determine for which values of x will $x^3 - 12x - 16 \leq 0$?

(f) Bepaal die vergelyking van die reguit lyn DA wat deur A gaan en loodreg op AB is.

(g) Bepaal die vergelyking van die reguit lyn BE wat deur B gaan en ewewydig aan DA is.

3. The absolute value function

3.1 Introduction

In this paragraph we investigate the properties of a function that is frequently used in function analysis. The absolute value function is practically used as a function that makes any number "positive", but it is also used to define the limit of a function in a more elegant way than the intuitive way it has been defined at school level.

3.2 Formal definition of the absolute value function

Complete the table:

x	$f(x) = \lvert x \rvert$	**Operation to make positive**	**Condition**
-3			
5			
x			
$x+2$			
x^2			

Formal definition of absolute value of a real number x:

$$\lvert x \rvert = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

Exercise

6. Use the formal definition of absolute value to solve:

(a) $\lvert x - a \rvert = 0$

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(b) $\lvert x - a \rvert = b$ and $b < 0$

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(c) $\lvert x - a \rvert = b$ and $b > 0$

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(d) $|x| \le a$ and $a > 0$

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(e) $|x| \ge a$ and $a > 0$

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(f) $|x - a| \le b$ and $b > 0$

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(g) $|x - a| \ge b$ and $b > 0$

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3.3 Properties of absolute value

$|x| = \sqrt{x^2}$

$|x - a|^2 = (x - a)^2$

$|a|\,|b| = |ab|$

$\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$

$|-x + a| = |x - a|$

If $|x - a| = 0$ then is $x = a$

If $|x - a| = b$ and $b > 0$, then $x = a + b$ or $x = a - b$

If $|x - a| \le b$ and $b > 0$, then $a - b \le x \le a + b$

If $|x - a| \ge b$ and $b > 0$, then is $x \le a - b$ or $x \ge a + b$

Exercise

7. Determine the values of x in the following equations:

(a) $\left| -x+2 \right| = 0$

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(b) $2\left| 2x+1 \right| = 6$

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(c) $\left| \frac{1}{2}x + \frac{1}{2} \right| = \left| -2 \right|$

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8. Solve for x and illustrate the solution on a real number line:

(a) $\left| x+2 \right| \leq 2$

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(b) $-\left| 2x-5 \right| < -3$

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(c) $2\left| -2x+3 \right| < 10 \cap -3 < x \leq 4$

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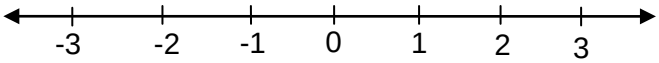
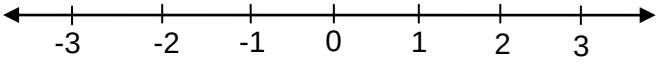
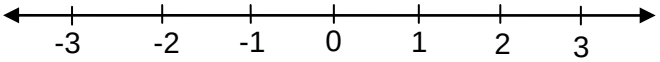
3.4 Absolute value as a distance



x	Distance from 0	$\lvert x \rvert = y$	x	Distance from 0	$\lvert x \rvert = y$
–3			1		
–2			2		
–1			3		
0					



x	Distance from 1	$\lvert x-1 \rvert = y$	x	Distance from 1	$\lvert x-1 \rvert = y$
–3			1		
–2			2		
–1			3		
0					



x	Distance from –1	$\lvert x+1 \rvert = y$	x	Distance from –1	$\lvert x+1 \rvert = y$
–3			1		
–2			2		
–1			3		
0					

Exercise

9. Use absolute value notation to express each of the following statements mathematically:

(a) x is less than 3 units from 7. .....

(b) x is no more than 5 units from 7. .....

(c) x is between –3 and 3. .....

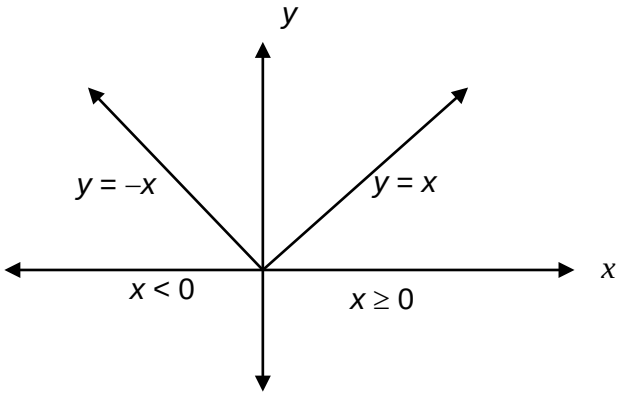
(d) The distance between 7 and x is 4. .....

(e) A interval of length *a* about *b*. .....

3.5 Graph of absolute value function

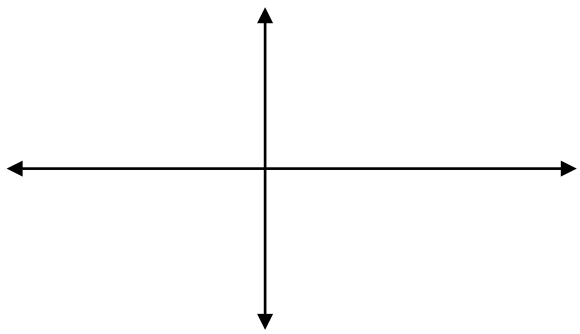
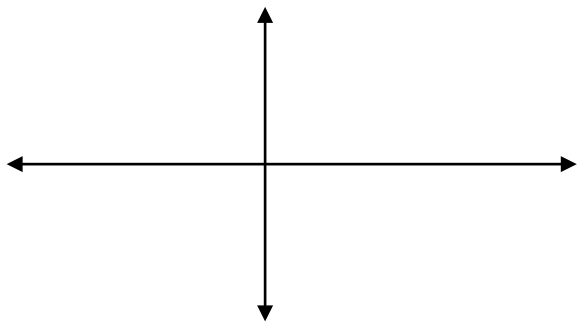
The basic form of the graph of the absolute value function can be deduced from application of the definition of absolute value, namely $|x| = \begin{cases} x, \ as \ x \geq 0 \\ -x, \ as \ x < 0 \end{cases}$.

It follows that the graph consist of two parts, the fisrt part is the straight line $y = x$ that is drawn for the interval where $x \geq 0$ and the second part is the straight line $y = -x$ that is drawn for the interval $x < 0$.

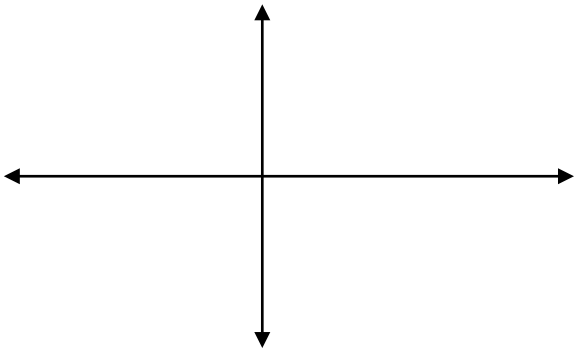
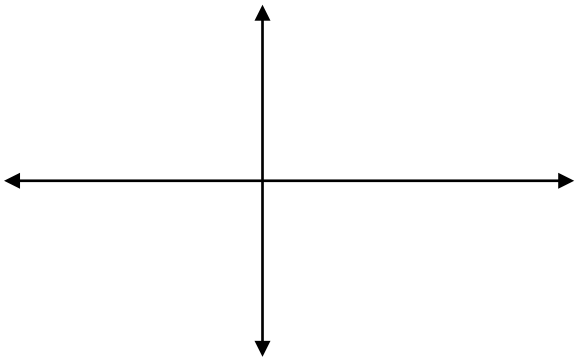


Exercise

10. Sketch the following graphs:

(a) $y = \|x - 1\| + 2$	
(b) $y = -\|2x + 2\|$	

11. Use sketch graphs to find solutions for the inequalities:

(a) $\lvert x+2 \rvert \leq 2$	
(b) $-\lvert 2x-5 \rvert < -3$	

4. Limits and continuity

4.1 The limit concept

Example 1:

Complete the table for the function $f(x) = x - 1$.

$x < 2$	$x > 2$
f (1) = ...	f (3) = ...
f (1,5) = ...	f (2,5) = ...
f (1,9) = ...	f (2,1) = ...
f (1,99) = ...	f (2,01) = ...
f (1,999) = ...	f (2,001) = ...

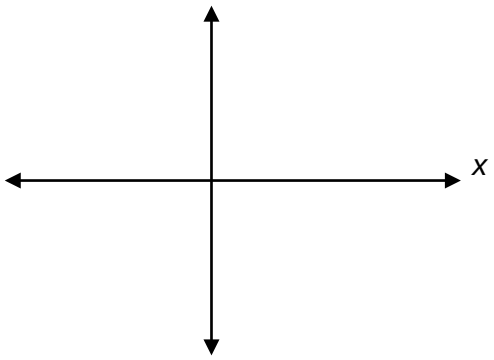
Conclusion from table:
As x approaches 2 from the left ($x < 2$), the function value approaches ……..
As x approaches 2 from the right ($x > 2$), the function value approaches …….
$\Rightarrow \quad \lim\limits_{x \to 2} f(x) =$ .....

NB: In example 1 the limit of the function is equal to the function value: $\lim\limits_{x \to 2} f(x) = 1 = f(2)$. It is not necessarily true that $\lim\limits_{x \to a} f(x) = f(a)$. If it is true, the function is continuous at that point.

Example 2:

Consider the function: $f(x) = \begin{cases} x - 1 & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$

Sketch the graph of this function on the system of axes given below:



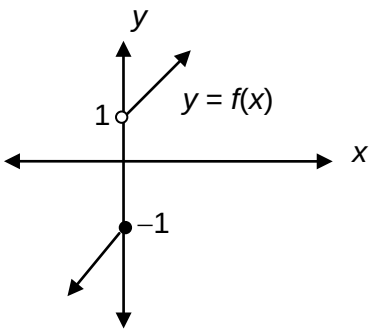
Conclusion from graph:
As x approaches 2 from the left ($x < 2$), the function value approaches ……..
As x approaches 2 from the right ($x > 2$), the function value approaches …..

$\Rightarrow \quad \lim\limits_{x \to 2} f(x) =$ .....

NB: $f(2) = 2$ and the limit is not equal to the function value.

Example 3:

Consider the function given in the sketch below:



Conclusion from graph:

The limit of $f(x)$ if x approaches 0 from the right (or positive side): $\lim\limits_{x \to 0^+} f(x) = ...$

The limit of $f(x)$ if x approaches 0 from the left (or negative side): $\lim\limits_{x \to 0^-} f(x) = ...$

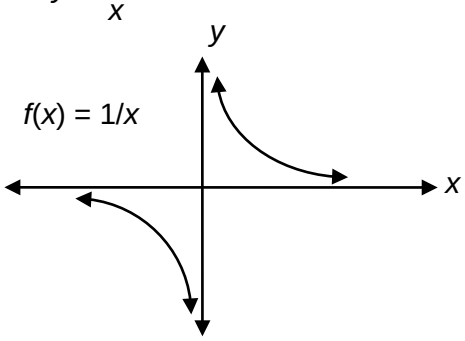
<u>NB</u>: The limit of $f(x)$ if x tends to 0 is not the same from the left-hand and the right-hand side. These limits are called **one-sided** limits.

Theorem:

The limit of f(x) as x approaches a exists if and only if **both** the **left-hand** and **right-hand** limits **exist** and if they are **equal**.

Example 4:

Consider the graph of function $y = \dfrac{1}{x}$



Conclusion from graph:

The limit of $f(x)$ if x approaches 0 from the right **does not** exist:

$$\lim_{x \to 0^+} f(x) = ...$$

The limit of $f(x)$ if x approaches 0 from the left **does not** exist:

$$\lim_{x \to 0^-} f(x) = ...$$

The limit of $f(x)$ if x approaches ∞ or $-\infty$ **does** exist:

$$\lim_{x \to \infty} f(x) = ... \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = ...$$

<u>NB</u>: The **limit** of the function exists as $x \to \infty$ or $x \to -\infty$, but the **function value** does not exist.

Definition of a limit:

Let f be a function defined on an open interval containing a, but f is not necessarily defined for a. The statement

$$\lim_{x \to a} f(x) = \ell$$

means that ℓ is the limit of the function $f(x)$ if $f(x)$ can be brought arbitrarily close to ℓ by choosing x close enough to a.

A more elegant way of defining a limit is as follows:
Let f be a function defined on an open interval containing a, but f is not necessarily defined for a. The statement

$$\lim_{x \to a} f(x) = \ell$$

means that for every $\in > 0$ there exists a $\delta > 0$ such that
if $0 < |x - a| < \delta$, then $|f(x) - \ell| < \in$.

This means that $f(x)$ will be as close to the number ℓ as one wants it to be, as long as x is sufficiently close to a.

<u>Some notes about limits</u>:

- The function value $f(a)$ does not have to be defined.
- The limit ℓ is not necessarily equal to $f(a)$.
- The limit is independent of the way in which $x \to a$, hence the limit should be the same whether x approaches a from the left or from the right.

Exercise

1. Determine the following limits (use the symbols ∞ or $-\infty$ when the limit does not exist):

(a) $\lim_{x \to -\infty} \frac{1}{x^2}$ = ...

(b) $\lim_{x \to 0^+} \frac{1}{x^2}$ = ...

(c) $\lim_{x \to 0^-} \frac{1}{x^2}$ = ...

(d) $\lim_{x \to \infty} \frac{1}{x^2}$ = ...

(e) $\lim_{x \to 2^-} \frac{x}{x-2}$ = ...

(f) $\lim_{x \to 2^+} \frac{x}{x-2}$ = ...

4.2 Limits and continuity

In this paragraph, the relationship between limits and continuity will be explored. The concept of continuity refers to a smooth curve or line that has no gaps or holes.

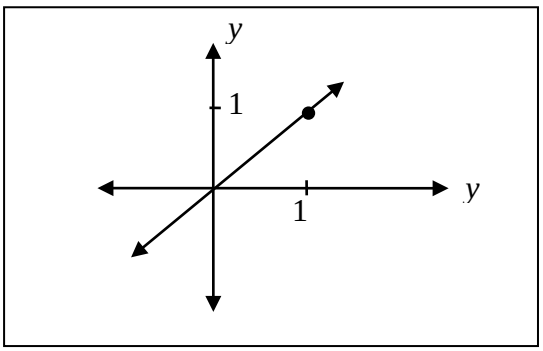
Example 1:

Consider the function: $f(x) = x$

$\lim_{x \to 1} f(x) = 1$

and $f(1) = 1$

$\Rightarrow \quad \lim_{x \to 1} f(x) = f(1)$



Example 2:

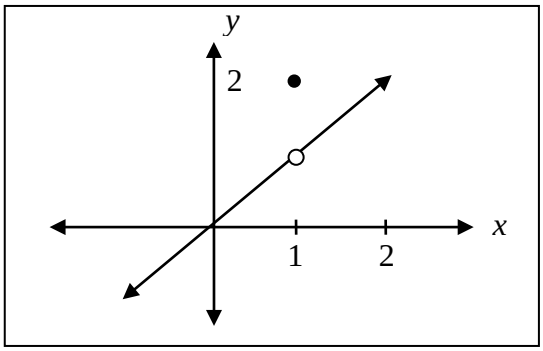
Consider the function:

$$g(x) = x \quad \text{if } x \neq 1,$$
$$= 2 \quad \text{if } x = 1$$

$$\lim_{x \to 1} g(x) = 1$$

but $g(1) = 2$

$$\Rightarrow \quad \lim_{x \to 1} g(x) \neq g(1)$$



Question: What is the difference between the graphs of example 1 and 2?

Answer : There is no breaks in the first graph. In the second graph there is a gap at the point where $x = 1$.

Definition:

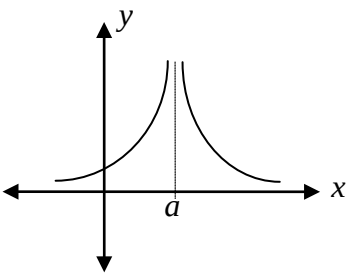
A function f is **continuous** at $x = a$ if and only if:

1. $f(x)$ is defined at $x = a$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Exercise

2. Examine all three conditions for continuity in order to explain why the following function are discontinuous at $x = a$:

(a)



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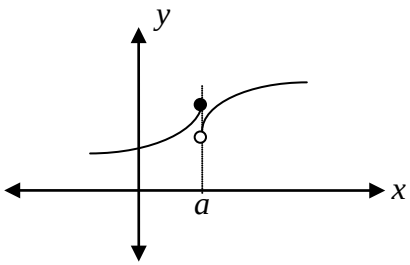
(b)



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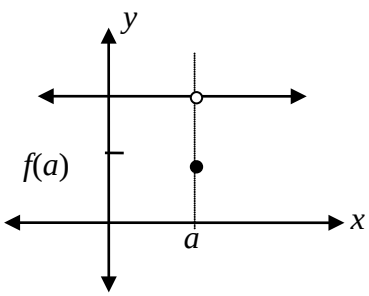
(c)



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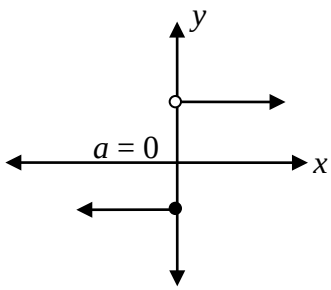
(d)



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3. Use the definition of continuity to show that the following function is continuous at the given point: $f(x) = \dfrac{x-3}{9x}$, $x = -3$

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4. Determine if the following function is continuous in the given points:

$$f(x) = \begin{cases} x+2, & \text{as } x \geq 2 \\ x, & \text{as } x < 2 \end{cases} \quad ; \quad x = 2, x = 0$$

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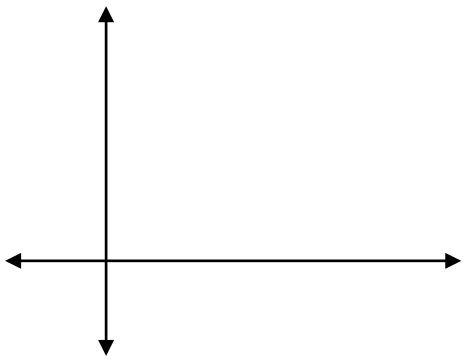


4.3 Properties of limits

The following properties of limits are given without formally proving them.

Rule 1: $\lim\limits_{x \to a} c = c$ where c is a constant.

Rule 2: $\lim\limits_{x \to a} x^n = a^n$ for any positive integer *a.*

Assuming that $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist:

Rule 3: $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

Rule 4: $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

Rule 5: $\lim\limits_{x \to a} [f(x)\, g(x)] = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)$

Rule 6: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, where $\lim\limits_{x \to a} g(x) \neq 0$

Rule 7: $\lim\limits_{x \to a} [c\, f(x)] = c \lim\limits_{x \to a} f(x)$ where *c* is a constant.

Rule 8: $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ if roots are defined.

Exercise

5. Determine the following limits:

(a) $\lim\limits_{x \to 1} 18$ = ...

(b) $\lim\limits_{x \to 2} x^3$ = ...

(c) $\lim\limits_{x \to 1} (x + 1)$ = ...

(d) $\lim\limits_{x \to 3} x^2$ =.....

(e) $\lim\limits_{x \to 1} \dfrac{x+1}{x}$ = ...

(f) $\lim\limits_{x \to 2} \sqrt{x^2}$ = ...

4.4 Limits of polynomials

Example: Determine the value of $\lim\limits_{x \to 3} (x^2 + x)$

$\lim\limits_{x \to 3} (x^2 + x) = \lim\limits_{x \to 3} x^2 + \lim\limits_{x \to 3} x$ (Rule 3)

$= (3)^2 + (3) = 12$ (Rule 2)

<u>NB</u>: It seems as if one can just **substitute** the value of *a* into the expression for *x* to find the limit. Substitution is a valid option when the function is a **polynomial function**.

Theorem:

If *f* is a **polynomial function**, then $\lim\limits_{x \to a} f(x) = f(a)$ for every real number *a*.

NB: It follows from the third condition of continuity that if $\lim\limits_{x \to a} f(x) = f(a)$ then the function is continuous. The inverse is also true: If a function is continuous, then $\lim\limits_{x \to a} f(x) = f(a)$

Exercise

6. Determine the following limits (use ∞ or $-\infty$ where applicable):

(a) $\lim\limits_{x \to 3} (x^2 + 4x - 5)$ = ...

(b) $\lim\limits_{x \to -3} 3x^3$ = ...

4.5 Limits of rational functions

Example: Determine $\lim\limits_{x \to 1} \dfrac{x}{x-1}$.

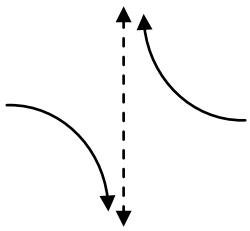
If one puts *x* equal to 1 the result is division by zero which is undefined. The limit can be determined informally as follows:

$x < 1$	$x > 1$
$f(0{,}5) = -1$ $f(0{,}9) = -9$ $f(0{,}99) = -99$ $f(0{,}999) = -999$ $f(0{,}999) = -9999$	$f(1{,}5) = 3$ $f(1{,}1) = 11$ $f(1{,}01) = 101$ $f(1{,}001) = 1001$ $f(1{,}0001) = 10001$

According to the pattern in the table, two one-sided limits can be determined:

$$\lim_{x \to 1^-} \frac{x}{x-1} = -\infty \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{x}{x-1} = \infty$$

This means an asymptote at $x = 1$ and the graph will look like this:



$x = 1$

Exercise

7. Determine the following limits (use the symbols ∞ or $-\infty$ where applicable):

(a) $\lim\limits_{x \to -2} \dfrac{x-2}{x+5} =$ .....

(b) $\lim\limits_{x \to -2^-} \dfrac{x}{x+2} =$ .....

(c) $\lim\limits_{x \to -2^+} \dfrac{x}{x+2} =$ .....

(d) $\lim\limits_{x \to 0^-} \sqrt{x} =$ .....

(e) $\lim\limits_{x \to 0^+} \sqrt{x} =$ .....

8. Does the following limits exist? Explain.

(a) $\lim\limits_{x \to -2} \dfrac{x}{x+2}$

....

.....

(b) $\lim\limits_{x \to 0} \sqrt{x}$

.....

4.6 The special case $\frac{0}{0}$

Example: Determine $\lim\limits_{x\to 1} \frac{x^2-1}{x-1}$.

Solution: $\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \frac{\lim\limits_{x\to 1}(x^2-1)}{\lim\limits_{x\to 1}(x-1)} = \frac{0}{0}$

This solution does not make sense. Analysing the function shows that $x - 1$ is a factor of $x^2 - 1$, hence one can first simplify the function before finding the Hierdie oplossing maak nie sin nie. Om 'n sinvolle antwoord te kry maak ons gebruik van die feit dat $x - 1$ 'n faktor is van $x^2 - 1$ om die uitdrukking te vereenvoudig:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to 1}(x+1) = 2$$

NB: Division by $(x - 1)$ is allowed when $x \to 1$ because $x - 1 \neq 0$

Exercise

9. Determine the following limits (use the symbols ∞ or $-\infty$ where applicable):

(a) $\lim\limits_{x\to 0} \frac{10x^2}{x}$ = ...

(b) $\lim\limits_{x\to -1} \frac{x^2-1}{x+1}$ = ...

(c) $\lim\limits_{h\to 0} \frac{|x+h|-|x|}{h}$ = ……..... ……..

……..... ……

……..... ……

.....

.....

4.7 The special case $\frac{\infty}{\infty}$

Example: Determine $\lim\limits_{x\to\infty} \frac{x^2}{x^2-1}$

Solution: $\lim\limits_{x\to\infty} \frac{x^2}{x^2-1} = \frac{\infty}{\infty}$

This solution does not make sense. In order to find a solution one has to get rid of all terms that have positive powers of x:

$$\lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{x\to\infty} \frac{1}{1-1/x^2} = \frac{1}{1-0} = 1$$ (÷ by x^2 in numerator and denominator)

NB: In the special case of ∞/∞, one first has to use algebraic manipulation to change all terms with positive powers of x to the form $1/x^n$ before using substitution. It follows that $\lim_{x\to\infty} x^n = 0$ for $n < 0$.

Exercise

10. Determine the following limits (use the symbols ∞ or –∞ where applicable):

(a) $\lim_{x\to-\infty} \frac{10x^2}{x}$ = ...

(b) $\lim_{x\to\infty} \frac{2x+5}{3x+2}$ = ...

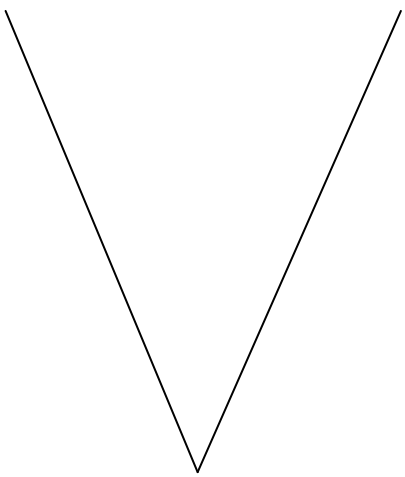
…….....

…….....

4.8 Limits of trigonometric functions

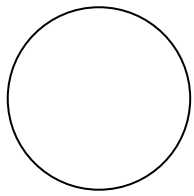
Definition of angle in radians

Use a protractor to measure the given angle in degrees.



Measure the arc length and the radius and determine the ratio of the arc length to radius. This ratio is the measure of an angle in radians.

Use the perimeter of a circle to determine a conversion factor between degrees and radians.



Exercise

11. Change to radians:

degrees	radians	degrees	radians	degrees	radians	degrees	radians
30^0		45^0		60^0		90^0	
150^0		135^0		120^0		180^0	
210^0		225^0		240^0		270^0	
330^0		315^0		300^0		360^0	

Limits of sine and cosine

The sinus and cosinus functions are continuous, hence their limits are equal to the function values:

Example 1: Determine $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\cos x}$

Solution: $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\cos x} = \dfrac{\cos 0 - 1}{\cos 0} = \dfrac{1-1}{1} = 0$

Example 2*:* Determine $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

In this case, substitution leads to a $\dfrac{0}{0}$ case. The solution can me determine intuitively by choosing the value near to 0 from the left and from the right:

$x < 0$	$x > 0$
(sin –1,0)/(–1,0) ~ 0,841	(sin 1,0)/(1,0) ~ 0,841
(sin –0,5)/(–0,5) ~ 0,959	(sin 0,5)/(0,5) ~ 0,959
(sin –0,1)/(–0,1) ~ 0,998	(sin 0,1)/(0,1) ~ 0,998
(sin –0,01)/(–0,01) ~ 0,99998	(sin 0,01)/(0,01) ~ 0,99998

$\Rightarrow \quad \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

LW: The angle must be in radians.

Example 3: Bepaal $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x}$

Solution: . $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = \lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} . \dfrac{\cos x + 1}{\cos x + 1}$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} = 1.\frac{0}{1+1} = 0$$

Exercise

12. Determine the following limits (use the symbols ∞ or $-\infty$ where applicable):

(a) $\lim\limits_{h\to 0} \dfrac{\sin 2h}{h}$ = ...

....

(b) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$ = ...

....

(c) $\lim\limits_{x\to \pi} \dfrac{\sin x}{x-\pi}$ = ...

....

(d) $\lim\limits_{t\to 0} \dfrac{1-\cos t}{\sin t}$ = ...

....

Limits to remember

$\lim\limits_{x\to a} f(x) = f(a)$ if $f(x)$ is continuous.

$\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = \mathrm{e}$

$\lim\limits_{h\to 0} \dfrac{a^h - 1}{h} = \log_{\mathrm{e}} a = \ln a$

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

$\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

5. Die derivative of a function

5.1 The relationship between derivative and slope of a graph

The derivative of a function gives the slope of tangent line to the graph of the function $f(x)$ at the point (x, y). The derivative can be determined as follows:

Consider the graph of function *f*(*x*) with tangent line t at point $P(x, y)$.

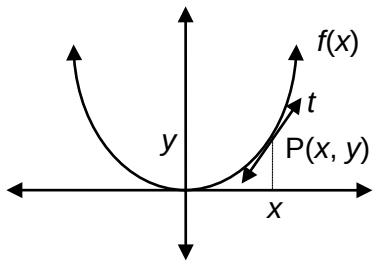


Figure 1

We would like to determine the slope/gradient of tangent line t, but one needs two points to determine the equation of a line.

Consider the graph of function *f*(*x*) with secant line PQ through the points $P(x, y)$ and $Q(x_1, y_1)$.

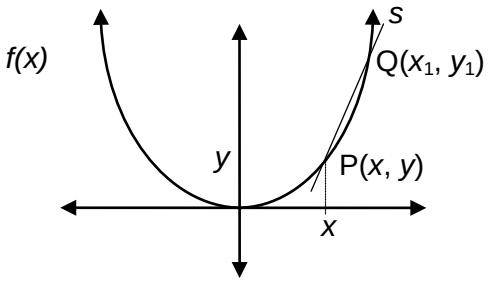
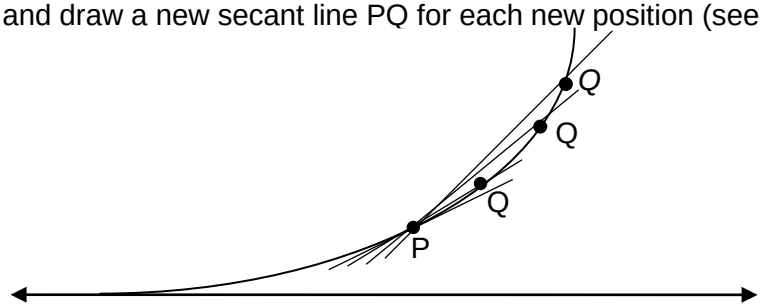


Figure 2

The slope of secant line s to the graph is given by: $m_{PQ} = \dfrac{y_Q - y_P}{x_Q - x_P}$.

To determine the slope at point P on the graph, we move point Q closer and closer to point P and draw a new secant line PQ for each new position (see fig 3).



Figuur 3

The slope at P can nou be seen as the limit of the slopes of all the secant lines:

$$m_P = \lim_{Q \to P} m_{PQ}$$

Applied to point P in figure 1 it means that:

$$m_P = \lim_{Q \to P} m_{PQ} = \lim_{Q \to P} \frac{f(Q) - f(P)}{Q - P} = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x}$$

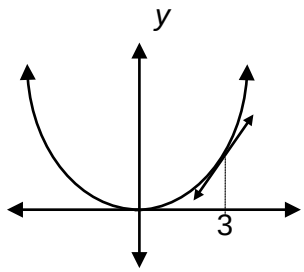
Let $x_1 - x = h$, then $x_1 = x + h$. Use the notation $f'(x)$ to indicate the slope of the tangent line at a point. In general the derivative of a function is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5.2 The derivative of a function from first principles

Example 1: Determine the derivative of $f(x) = x^2$ at $x = 3$ and $x = -3$.

Prediction: Consider the graph of $f(x) = x^2$:



The slope of the tangent line at $x = 3$
Must be positive; and the slope of
The tangent line at $x = -3$ must be negative.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

$\Rightarrow$ $f'(-3) = 2(-3) = -6$, which is negative, and $f'(3) = 2(3) = 6$, which is positive.

Example 2: Determine the derivative of $y = \sin x$ from first principles.

Solution:

$$D_x \sin x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h} = \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \,.\, 0 + \cos x \,.1 = \cos x$$

Example 3: Differentiate from first principles: $y = e^x$.

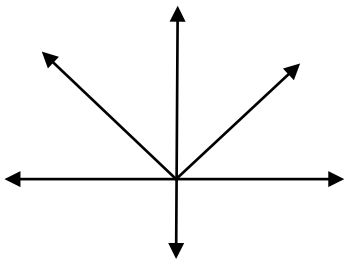
Solution: $$D_x e^x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right) = e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = e^x (\ln e) = e^x$$

NB: The natural exponential function is its own derivative.

Example 4: Determine the derivative of $y = |x|$

Solution: Consider the graph of $y = |x|$.



Conclusion from graph:

One cannot draw a unique tangent to the graph at $x = 0$, therefore the function is not differentiable at $x = 0$.

Differentiate the function:

$$\begin{aligned}
D_x\,|x| &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{|x+h|-|x|}{h} \\
&= \lim_{h\to 0}\left(\frac{|x+h|-|x|}{h}\right)\left(\frac{|x+h|+|x|}{|x+h|+|x|}\right) \\
&= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h(|x+h|+|x|)} = \lim_{h\to 0}\frac{2xh+h^2}{h(|x+h|+|x|)} \\
&= \lim_{h\to 0}\frac{2x+h}{|x+h|+|x|} = \frac{2x}{2|x|} = \frac{x}{|x|}
\end{aligned}$$

NB: The graph of $y = |x|$ is continuous in every point, but it is not differe differentiable in every point.

This observation leads to the formulation of the following theorem:

Theorem:

If f is differentiable at $x = a$, then f is continuous at a.

<u>NB</u>: The **converse** is not necessarily true.

Exercise

1. Determine the derivative of the following functions from first principles:

(a) $y = |2x|$

……..... ……

……..... ……

.....

.....

.....

.....

.....

.....

(b) $y = \cos x$

……..... ……

……..... ……

.....

.....

5.3 Rules of differentiation

It is time consuming to differentiate from first principles. There are various rules that can speed up the process. These rules are derived using the formula $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ and the rules of limits.

Constants: D_x (constant) = 0

Addition: $D_x[f(x)+g(x)] = D_x f(x) + D_x g(x)$

Subtraction: $D_x[f(x)-g(x)] = D_x f(x) - D_x g(x)$

Scalar multplication: $D_x[cf(x)] = cD_x f(x)$

Standard derivative : $D_x x^n = nx^{n-1}$ with n a rational number.

Product rule: $D_x[f(x)g(x)] = f(x)D_x g(x) + g(x)D_x f(x)$

Quotient rule: $D_x \frac{f(x)}{g(x)} = \frac{g(x)\ D_x f(x) - f(x)\ D_x g(x)}{(g(x))^2}$, with $g(x) \neq 0$.

Chain rule: $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}$, with $y = f(v)$; $v = g(u)$, $u = h(x)$

Trigonometric functions:

$D_x \sin x = \cos x$ $\quad$ $D_x \cos x = -\sin x$ $\quad$ $D_x \tan = \sec^2 x$

$D_x \csc x = -\csc x \cot x$ $\quad$ $D_x \sec x = \sec x \tan x$ $\quad$ $D_x \cot x = -\csc^2 x$

Exponential functions

$D_x e^x = e^x$ $\quad$ $D_x a^x = a^x(\ln a)$ $\quad$ $D_x e^u = e^u D_x u$

Logarithmic functions:

$D_x \ln x = \frac{1}{x}$ $\quad$ $D_x \log_a x = \frac{1}{x}(\log_a e)$ $\quad$ $D_x \ln u = \frac{1}{u} D_x u$

Absolute value function: $D_x |x| = \frac{x}{|x|}$

Exercise

2. Determine the average gradient of $f(x) = x^2 + 2$ between $x = 3$ and $x = 5$.

……..... ……

……..... ……

.....

3. If $f(x) = 3x^2 + 2$, determine the slope of the tangent line to the graph of f at the point $x = -2$.

……..... ……

……..... ……

4. Differentiate with respect to x:

(a) $f(x) = 4x^3 + x\sqrt{x} - \dfrac{5}{x^4}$

……..... ……

……..... ……

……..... ……

(b) $f(x) = (2x + 1)^2$

……..... ……

……..... ……

……..... ……

(c) $f(x) = \sqrt[4]{4x^2 - 6x}$

……..... ……

……..... ……

……..... ……

(d) $f(x) = (2x^2 + 4x)^{100}$

……..... ……

……..... ……

……..... ……

(e) $f(x) = (x^2 + 6x)(x^3 - 6x^2)$

……

……

……

(f) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

……

……

……

……

(g) $f(x) = \dfrac{x^2 + 3}{5}$

……

……

……

……

(h) $f(x) = \sin x \cos x$

……

……

……

(i) $f(x) = \sin x^2$

……

……

……

(j) $f(x) = \cos 3x$

……

……

……

(k) $f(x) = \dfrac{\sin x}{x}$

……..... ……

……..... ……

……..... ……

(l) $f(x) = \sin(\cos x)$

……..... ……

……..... ……

……..... ……

(m) $f(x) = e^{x^2+1}$

……..... ……

……..... ……

……..... ……

(n) $f(x) = x^2 e^x$

……..... ……

……..... ……

……..... ……

(o) $f(x) = e^{-x} \ln x$

……..... ……

……..... ……

……..... ……

5.4 Graphs of higher order polynomials

The graphs of polynomials follow the pattern:

Polynomial of the first degree: $y = ax + b$

Graph is straight line with one $x-$intercept an no turning point.

Polynomial of the second degree: $y = ax^2 + bx + c$

Graph is a parabola with two $x-$intercepts and one turning point.

Polynomial of the third degree: $y = ax^3 + bx^2 + cx + d$

Graph is a curve with three $x-$intercepts and two turning points.

Polynomial of the fourth degree: $y = ax^4 + bx^3 + cx^2 + dx + e$

Graph is a curve with four $x-$intercepts and three turning points.

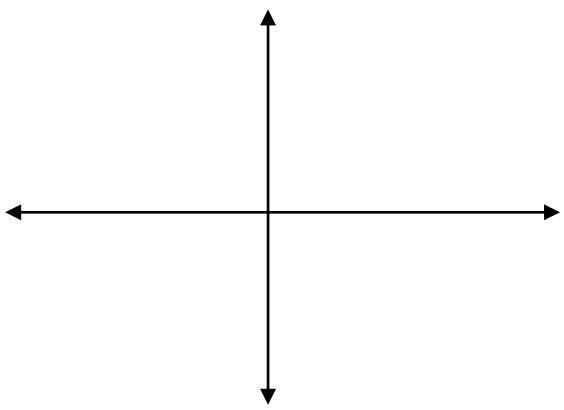
The following method can be used to determine the graph of a polynomial:

- Find ***y*–intercepts** by calculating $y = f(0)$.
- Find ***x*– intercepts** by calculating the value(s) of x if $f(x) = 0$.
- Find the **turning points** by determining the first derivative of the function. Put the first derivative equal to zero and solve for x to find the x–coordinates of the turning points. Substitute x into $f(x)$ to find the y–coordinate of the turning point.
- To discriminate between maximum and minimum turning points, determine the second derivative. If $f''(x) > 0$, then x is a minimum turning point, if $f''(x) < 0$, x is a maximum turning point and if $f''(x) = 0$, it could be a point of inflection.

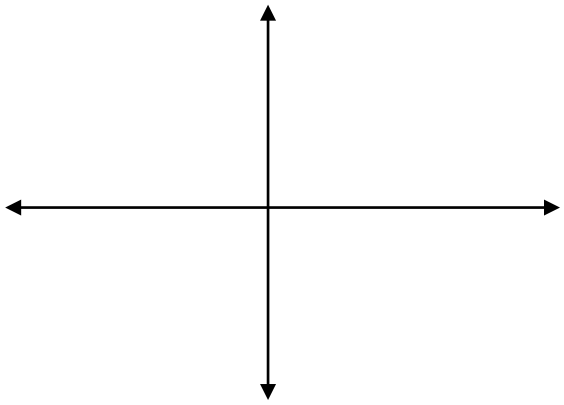
Exercise

5. Sketch graphs of the following polynomials:

(a) $y = -x^3 + 5x^2 + 8x - 12$



(b) $y = x^2(x^2 - 8)$.



5.5 Graphs of rational functions

Rational functions are characterised by asymptotes. The following method is advised for finding the graphs of rational functions:

- Determine the asymptotes of the graph using the following limits

> The line $x = a$ is a **vertical asymptote** of the graph of the function f if and only if
>
> $$\lim_{x \to a^+} f(x) = \infty \text{ (or } -\infty) \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \infty \text{ (or } -\infty)$$

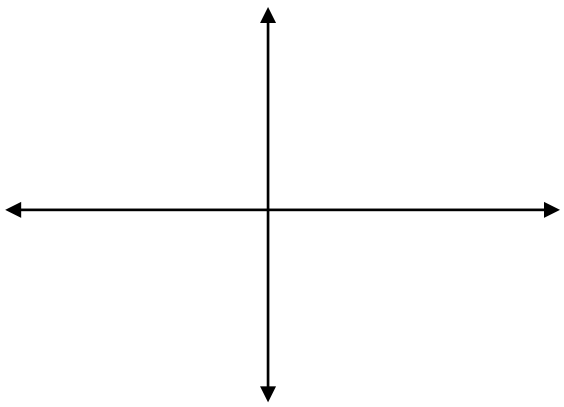
> The line $y = b$ is a **horizontal asymptote** of the graph of the function f if and only if
>
> $$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

- Find ***y*–intercepts** using $y = f(0)$.
- Find ***x*–intercepts** using $f(x) = 0$.
- If applicable, find **turning points** as described above.

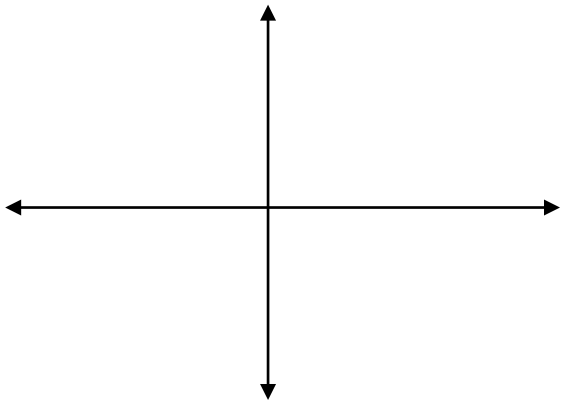
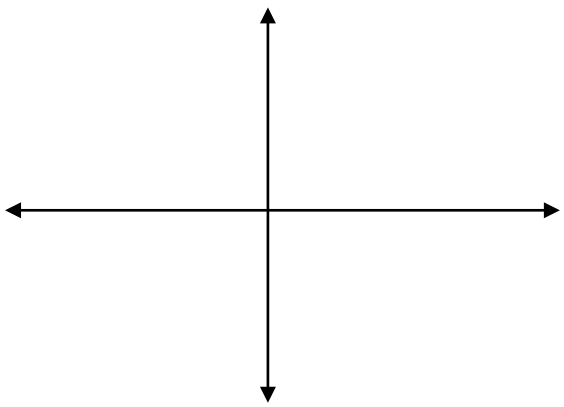
Exercise

6. Sketch the graphs of the following rational functions:

(a) $y = \dfrac{x}{x+1}$



(b) $y = \dfrac{4x}{x^2 + 1}$



(c) $y = \dfrac{x^2 - 2x}{x^2 - 4}$